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Vortex stretching and reconnection in a compressible fluid

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Abstract. Vortex stretching in a compressible fluid is considered. Two-dimensional (2D) and axisymmetric cases are considered separately. The flows associated with the vortices are *perpendicular* to the plane of the uniform straining flows. Externally-imposed density build-up near the axis leads to enhanced compactness of the vortices — "dressed" vortices (in analogy to "dressed" charged particles in a dielectric system). The compressible vortex flow solutions in the 2D as well as axisymmetric cases identify a length scale relevant for the compressible case which leads to the Kadomtsev-Petviashvili spectrum for compressible turbulence. Vortex reconnection process in a compressible fluid is shown to be possible even in the inviscid case — compressibility leads to defreezing of vortex lines in the fluid.

PACS. 47.32.-y Vortex dynamics; rotating fluids

1 Introduction

The vortex stretching process

- leads to the transport of energy among various scales of motion in a turbulent flow;
- plays an important role in the vortex reconnection process and hence in describing the fine scales of turbulence.

Vortex reconnection (Siggia and Pumir [1], Schatzle [2]) has been argued to be a prime candidate for a *finite-time* singularity in Navier-Stokes equations. Such a singularity plays a central role in the small-scale dynamics of turbulence by producing arbitrarily large velocity gradients. However, vortex reconnection is a process that is not yet well understood. Certain "canonical" cases of vortex reconnection have been investigated in great detail, both experimentally (Fohl and Turner [3], Oshima and Asaka [4]) and numerically (Ashurst and Meiron [5], Pumir and Kerr [6], Kida and Takaoka [7] and others). But, a *global* view of the various reconnection scenarios is not at hand yet.

Laboratory Experiments (Cadot et al. [8], Villermaux et al. [9]) and DNS (Jimenez et al. [10]) have revealed strong coherent and elongated vortices among the small scales in incompressible turbulence. These vortices are believed to originate from strained vorticity fields like the *Burgers* vortex (Burgers [11]). Burgers vortex describes the interplay between the intensification of vorticity due to the imposed straining flow and the diffusion of vorticity due to the action of viscosity. The straining simulates *locally* the stretching undergone by each vortex in the velocity field induced by other vortices. Intermittency structures that exhibit velocity profiles similar to that of Burgers vortex have been observed in grid turbulence (Mouri et al. [12]).

The two-dimensional (2D) Burgers vortex solution is of the form (Robinson and Saffman [13])

$$\mathbf{v} = \{-\alpha x + u(x,y,t), -\beta y + v(x,y,t), (\alpha+\beta)z\}. \quad (1)$$

The quantity $(\alpha+\beta)$ (α and $\beta > 0$) measures the stretching rate of vortices, which are aligned along the z-axis (that is also the principal axis of a uniform plane straining flow). Numerical solutions of three-dimensional (3D) Navier-Stokes equations (Ashurst et al. [14] and others) have confirmed the alignment between the vorticity and one principal axis of the local strain. The velocity induced by the vorticity lies in the xy-plane, with components u and v which are independent of z. Simple closed-form steady solutions exist for the following special cases

- $\alpha = \beta > 0$ - axisymmetric vortex;
- $\alpha > 0, \beta = 0$ - 2D shear layer.

Robinson and Saffman [13] demonstrated the existence of solutions for arbitrary values of the ratio α/β.

Unsteady 2D Burgers vortex solutions have been used to model the spanwise structure of turbulent mixing layers (Lin and Corcos [15], Neu [16]). Unsteady axisymmetric Burgers vortex solutions have been used to model the fine-scale structure of homogeneous incompressible turbulence (Townsend [17], Lundgren [18]).

DNS (Porter et al. [19]) have confirmed the existence of vortex filaments in compressible turbulence. The vortex stretching process can be expected to be influenced in an essential way by fluid compressibility (Shivamoggi [20]

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and [21]). So, investigation of stretched vortices in a compressible fluid is in order which is addressed in this paper along with applications to compressible turbulence.

Vortex reconnection in a compressible fluid is a topic in its infancy (Virk et al. [22] and Shivamoggi [23]). Additional mechanisms of vorticity generation like the *baroclinic* vorticity generation exist in a compressible fluid[1]. The vortex reconnection process in a compressible fluid is therefore more complicated than its counterpart in an incompressible fluid. Further exploration of the basic mechanism underlying this process is in order and is addressed in general terms in this paper.

2 Modified 2D Burgers vortex

Consider a modified Burgers vortex flow with the velocity field given by

$$\mathbf{v} = \{-\gamma(t)x, \gamma(t)y, W(x,t)\}. \tag{2}$$

(2) describes the convection of the vortex lines toward the y-axis and the stretching along the y-axis by the imposed straining flow. The straining flow is externally imposed, so the vorticity is decoupled from the dynamics of the straining flow that is stretching it. The streamlines (see Fig. 1) shown in the x, y-plane represent the uniform plane straining flow. This streamline pattern is the same in each plane parallel to the x, y-plane. Observe that the flow associated with the vortex in question is *perpendicular* to the plane of the uniform straining flow, unlike the Burgers vortex given by (1). This situation is well-suited for modelling a mixing-layer flow or jet flow. (2) describes the convection of the vortex lines towards the $x = 0$ plane and the stretching in the y-direction by the imposed straining flow. The vorticity field corresponding to (2) is

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \left\{0, -\frac{\partial W}{\partial x}, 0\right\} \tag{3}$$

which shows that the vortex lines for this model are aligned along the y-axis which happens to be the principal axis of the uniform plane straining flow (2), however, as in the Burgers vortex model (1).

Using (2) and (3), the vorticity conservation equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega} \tag{4}$$

becomes

$$\frac{\partial \Omega}{\partial t} - \gamma x \frac{\partial \Omega}{\partial x} = \gamma\Omega + \nu\frac{\partial^2 \Omega}{\partial x^2} \tag{5}$$

where ν is the kinematic viscosity and Ω is the vorticity-

$$\Omega \equiv \frac{\partial W}{\partial x}. \tag{6}$$

[1] See Samtaney et al. [24] for an explicit calculation for a planar gas interface refracting a shock.

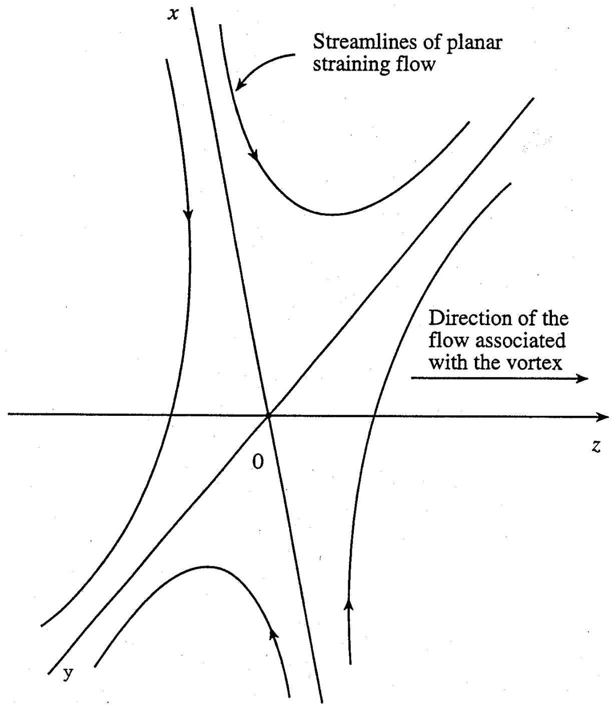

Fig. 1. Modified Burgers vortex model.

Introducing dimensionless independent variables

$$\xi = \sqrt{\frac{\gamma}{\nu}}\, x, \quad \tau = \int^{t} \gamma(t')dt' \tag{7}$$

equation (5) becomes

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \Omega}{\partial \xi} + \xi\Omega\right) = \frac{\partial \Omega}{\partial \tau}. \tag{8}$$

Let the boundary conditions be

$$|\xi| \Rightarrow \infty : \Omega \Rightarrow 0. \tag{9}$$

(i) Steady case:

For the steady case (with γ = constant), equation (8) becomes

$$\frac{d}{d\xi}\left(\frac{d\Omega}{d\xi} + \xi\Omega\right) = 0. \tag{10}$$

Using (9), equation (10) has the solution

$$\Omega = c_1 e^{-\xi^2/2} \tag{11}$$

or

$$W(\xi) = c_1 erf\left(\xi/\sqrt{2}\right) \tag{12}$$

which represents the shear layer.

For this shear-layer flow solution, the build-up of vorticity due to the convection of the vortex lines towards the $x = 0$ plane and the stretching in the y-direction by the imposed straining flow is counterbalanced by the diffusion of vorticity in the x-direction.

(ii) Unsteady case:

For the unsteady case, let us look for a solution of the form

$$\Omega(\xi,\tau) = h_\lambda(\xi)e^{-\lambda\tau}. \tag{13}$$

Equation (8) then yields

$$\frac{d}{d\xi}\left(\frac{dh_\lambda}{d\xi} + \xi h_\lambda\right) = -\lambda h_\lambda. \tag{14}$$

For bounded solutions of equation (14) to exist, we require

$$\lambda = n; \quad n = 0, 1, 2, \dots \tag{15}$$

Equation (14) then has the solution

$$h_n(\xi) = (-1)^n h_0(\xi) H_n(\xi); \quad n = 0, 1, 2, \dots \tag{16}$$

where,

$$h_0(\xi) = e^{-\xi^2/2}$$

and $H_n(\xi)$ are the Hermite polynomials

$$H_0(\xi) = 1, \;\; H_1(\xi) = \xi, \;\; H_2(\xi) = \xi^2 - 1, \;\; H_3(\xi) = \xi^3 - 3\xi, \;\; \text{etc.}$$

Observe that $n = 0$ (steady case) corresponds to the shear-layer solution (11) while $n = 1$ (unsteady case) corresponds to the jet solution.

3 Compressible modified 2D Burgers vortex

Let us now consider the modified 2D Burgers vortex in a compressible barotropic fluid. For this purpose, let the velocity and density profiles be given by (Shivamoggi [20])

$$\mathbf{v} = \{\dot\alpha(t)x, \dot\beta(t)y, W(x,t)\} \tag{17a}$$

$$\rho = \sigma(t) + \frac{\rho_0}{U}(\dot\alpha + \dot\beta)x. \tag{17b}$$

where ρ_0 and U are reference density and velocity, respectively. Equation (17) describes a density build-up (or decrease) in the direction along which vortex lines are being compressed by the imposed straining flow. This arrangement maximizes compressibility effects on the vortex stretching process.

Using (17), the mass-conservation equation

$$\frac{\partial\rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho + \rho(\nabla\cdot\mathbf{v}) = 0 \tag{18}$$

yields

$$\dot\sigma + \frac{\rho_0}{U}\left(\ddot\alpha + \ddot\beta\right)x + \dot\alpha\frac{\rho_0}{U}x(\ddot\alpha + \ddot\beta) + \left[\sigma + \frac{\rho_0}{U}(\dot\alpha + \dot\beta)x\right](\dot\alpha + \dot\beta) = 0 \tag{19}$$

from which, we obtain the following relations

$$\dot\sigma + \sigma(\dot\alpha + \dot\beta) = 0 \tag{20}$$

$$(\ddot\alpha + \ddot\beta) + \dot\alpha(\dot\alpha + \dot\beta) + (\dot\alpha + \dot\beta)^2 = 0. \tag{21}$$

Using equation (20), equation (21) becomes

$$\frac{d}{dt}\left(\frac{\dot\sigma}{\sigma}\right) + \dot\alpha\left(\frac{\dot\sigma}{\sigma}\right) - \left(\frac{\dot\sigma}{\sigma}\right)^2 = 0. \tag{22}$$

Next, using (17), the vorticity conservation equation

$$\rho\left[\frac{\partial\boldsymbol\omega}{\partial t} + (\mathbf{v}\cdot\nabla)\boldsymbol\omega\right] + \nabla\rho\times\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = \rho(\boldsymbol\omega\cdot\nabla)\mathbf{v} - \rho(\nabla\cdot\mathbf{v})\boldsymbol\omega + \mu\nabla^2\boldsymbol\omega \tag{23}$$

leads to

$$\left[\sigma + \frac{\rho_0}{U}(\dot\alpha + \dot\beta)x\right]\left[\frac{\partial^2 W}{\partial x\partial t} + \dot\alpha x\frac{\partial^2 W}{\partial x^2}\right] + \frac{\rho_0}{U}(\dot\alpha + \dot\beta)\left[\frac{\partial W}{\partial t} + \dot\alpha x\frac{\partial W}{\partial x}\right] = -\dot\alpha\frac{\partial W}{\partial x}\left[\sigma + \frac{\rho_0}{U}(\dot\alpha + \dot\beta)x\right] + \mu\frac{\partial^3 W}{\partial x^3} \tag{24}$$

$$(\dot\alpha + \dot\beta)(\ddot\beta + \dot\beta^2) = 0 \tag{25}$$

where μ is the dynamic viscosity.

Using (20), and the z-component of the equation of motion, namely,

$$\rho\left(\frac{\partial W}{\partial t} + \dot\alpha x\frac{\partial W}{\partial x}\right) = \mu\frac{\partial^2 W}{\partial x^2} \tag{26}$$

equations (24) and (25) become

$$\left[\sigma - \frac{\rho_0}{U}\left(\frac{\dot\sigma}{\sigma}\right)x\right]\left[\frac{\partial\Omega}{\partial t} + \dot\alpha x\frac{\partial\Omega}{\partial x} + \dot\alpha\Omega\right] - \frac{\frac{\rho_0}{U}\left(\frac{\dot\sigma}{\sigma}\right)}{\left[\sigma - \frac{\rho_0}{U}\left(\frac{\dot\sigma}{\sigma}\right)x\right]}\mu\frac{\partial\Omega}{\partial x} = \mu\frac{\partial^2\Omega}{\partial x^2} \tag{27}$$

$$\frac{\dot\sigma}{\sigma}\left(\ddot\beta + \dot\beta^2\right) = 0. \tag{28}$$

For the compressible case, $\dot\sigma/\sigma \neq 0$, so equation (28) reduces to

$$\ddot\beta + \dot\beta^2 = 0. \tag{29}$$

In order to facilitate an analytic solution, consider the case

$$\sigma = e^{\int^t c(t')dt'}, \quad \dot\alpha = -a, \quad \dot\beta = b. \tag{30}$$

Equations (20), (22) and (29) then yield a closed set of equations for $a(t), b(t)$ and $c(t)$:

$$b(t) = \frac{1}{t+A}, \quad c(t) = \frac{e^{\int^t a(t')dt'}}{B - \int^t e^{\int^{t'} a(t'')dt''}dt'}, \quad a(t) = c(t) + b(t), \tag{31}$$

where A and B are arbitrary constants.

Equation (27) then becomes

$$\left(\rho_0 e^{\int^t c(t')dt'} - \frac{\rho_0}{U} cx\right)\left(\frac{\partial\Omega}{\partial t} - ax\frac{\partial\Omega}{\partial x} - a\Omega\right) - \frac{\left(\frac{\rho_0}{U}\right)c}{\left(\rho_0 e^{\int^t c(t')dt'} - \frac{\rho_0}{U}cx\right)}\mu\frac{\partial\Omega}{\partial x} = \mu\frac{\partial^2\Omega}{\partial x^2}. \quad (32)$$

In order to simplify equation (32), let us assume that the compressibility effects are weak. From (20) and (30), this implies that the quantity $c(t)$, which is a measure of the density change, is small. Equation (22) then leads to

$$\frac{dc}{dt} - a(t)c \approx 0 \quad (33)$$

from which,

$$c(t) \approx c_0 e^{\int^t a(t')dt'} \quad (34)$$

c_0 being an arbitrary constant. Equation (34) replaces the second of the three solutions in (31). Further, in the weakly-compressible case, equation (28) (which, to first approximation, is automatically satisfied) does not lead to equation (29) which, therefore, has to be abandoned. This implies that the first of the three solutions in (31), which comes from equation (29), also has to be dropped.

Thus, keeping only terms of $O(c)$, and introducing dimensionless independent variables

$$\xi \equiv \sqrt{\frac{a}{\mu}}\, x, \quad \tau \approx \int^t a(t')dt' \quad (35)$$

equation (32) may be approximated by

$$\frac{\partial}{\partial\xi}\left[\frac{\partial\Omega}{\partial\xi} + \xi\Omega\right] = \frac{\partial\Omega}{\partial\tau} - \tilde{c}\frac{\partial\Omega}{\partial\xi} \quad (36)$$

where,

$$\tilde{c}(\tau) \equiv \frac{c(t(\tau))}{\rho_0 U}\sqrt{\frac{\mu}{a(\tau)}}.$$

The approximation implicit in equation (36) gets better as viscosity increases. The boundary conditions on Ω are the same as in (9).

Comparison of equation (36) with the corresponding equation (8) for the incompressible case shows that the last term on the right hand side in equation (36) represents the *first*-order contribution due to the compressibility effects (assumed to be weak). Further, observe that the compressibility effects impart hyperbolic character to equation (36), associated with sound-wave propagation in the fluid.

As a first approximation, if we ignore the time-dependence of the straining-flow profiles, and hence, $\tilde{c}(\tau)$, and treat $\tilde{c}(\tau)$ as a constant, equation (36) can be solved exactly to give

$$\Omega(\tau,\xi) \approx (-1)^n e^{-\frac{(\xi+\tilde{c}\tau)^2}{2}} H_n(\xi + \tilde{c}\tau)e^{-n\tau} \quad (37)$$

where $H_n(\xi)$ are the *Hermite* polynomials.

Comparison of the compressible vortex profile (37) with the corresponding compressible vortex profile (13), (15) and (16) shows that, for the 2D case, the *first*-order effect of compressibility is to cause a mere *Galilean* translation in space of the incompressible vortex profiles. Therefore, in order to capture non-trivial effects of compressibility in the 2D case one needs to consider the time dependence of the straining-flow profiles. This restriction turns out to be relaxed for the axisymmetric case (below).

4 Compressible axisymmetric stretched vortex

Consider an axisymmetric stretched vortex in a compressible barotropic fluid (Shivamoggi [21]). Let the velocity (in cylindrical polar coordinates (r,θ,z)) and the density profiles be given by[2]

$$\mathbf{v} = \{\dot{\alpha}(t)r, W(r,t), \dot{\beta}(t)z\} \quad (38a)$$

$$\rho = \sigma(t) + \frac{\rho_0}{U}(2\dot{\alpha} + \dot{\beta})r \quad (38b)$$

where ρ_0 and U are reference density and velocity, respectively. (38) describes a density build-up (or decay) towards the z-axis (which is also the direction along which vortex lines are being compressed by the imposed straining flow).

The vorticity field corresponding to (38) is

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \{0, 0, \Omega\}, \quad (39)$$

where

$$\Omega = D_r W, \quad D_r \equiv \frac{\partial}{\partial r} + \frac{1}{r}.$$

Equation (39) shows that the vortex lines for this model are aligned along the z-axis, which happens to be the principal axis of the axisymmetric uniform straining flow (38). Further, the flow associated with the vortex is again perpendicular to the plane of the uniform straining flow, a situation that is well suited to modeling an axisymmetric mixing-layer flow. Equation (38) describes the convection of the vortex lines towards the z-axis and the stretching along the z-axis by the imposed straining flow.

Using (38), the mass-conservation equation (18) yields

$$\dot{\sigma} + \frac{\rho_0}{U}\left(2\ddot{\alpha} + \ddot{\beta}\right)r + \dot{\alpha}r\frac{\rho_0}{U}\left(2\dot{\alpha} + \dot{\beta}\right) + \left[\sigma + \frac{\rho_0}{U}\left(2\dot{\alpha} + \dot{\beta}\right)r\right]\left(2\dot{\alpha} + \dot{\beta}\right) = 0 \quad (40)$$

which leads to the following relations

$$\dot{\sigma} + \sigma(2\dot{\alpha} + \dot{\beta}) = 0 \quad (41)$$

$$(2\ddot{\alpha} + \ddot{\beta}) + \dot{\alpha}(2\dot{\alpha} + \dot{\beta}) + (2\dot{\alpha} + \dot{\beta})^2 = 0. \quad (42)$$

Combining equations (41) and (42), we obtain

$$\frac{d}{dt}\left(\frac{\dot{\sigma}}{\sigma}\right) + \dot{\alpha}\left(\frac{\dot{\sigma}}{\sigma}\right) - \left(\frac{\dot{\sigma}}{\sigma}\right)^2 = 0. \quad (43)$$

[2] A general class of velocity-field profiles of which (38a) is a special case has been discussed by Ohkitani and Gibbon [25].

Next, using (38), the vorticity conservation equation (23) leads to

$$\left[\sigma + \frac{\rho_0}{U}\left(2\dot{\alpha} + \dot{\beta}\right) r\right]\left[\frac{\partial \Omega}{\partial t} + \dot{\alpha} r \frac{\partial \Omega}{\partial r}\right] + \frac{\rho_0}{U}\left(2\dot{\alpha} + \dot{\beta}\right)\left[\frac{\partial W}{\partial t} + \dot{\alpha} r \frac{\partial W}{\partial r} + \dot{\alpha} W\right] = -2\left[\sigma + \frac{\rho_0}{U}\left(2\dot{\alpha} + \dot{\beta}\right) r\right]\dot{\alpha}\Omega + \mu D_r \frac{\partial \Omega}{\partial r} \quad (44)$$

$$\left(2\dot{\alpha} + \dot{\beta}\right)\left(\ddot{\beta} + \dot{\beta}^2\right) = 0. \quad (45)$$

Using equation (41), and the θ-component of the equation of motion, namely

$$\rho\left(\frac{\partial W}{\partial t} + \dot{\alpha} r \frac{\partial W}{\partial r} + \dot{\alpha} W\right) = \mu \frac{\partial}{\partial r}(D_r W) \quad (46)$$

equations (44) and (45) become

$$\left[\sigma - \frac{\rho_0}{U}\left(\frac{\dot{\sigma}}{\sigma}\right) r\right]\left[\frac{\partial \Omega}{\partial t} + \dot{\alpha} r \frac{\partial \Omega}{\partial r} + 2\dot{\alpha}\Omega\right] - \frac{\frac{\rho_0}{U}\left(\frac{\dot{\sigma}}{\sigma}\right)}{\left[\sigma - \frac{\rho_0}{U}\left(\frac{\dot{\sigma}}{\sigma}\right) r\right]}\mu \frac{\partial \Omega}{\partial r} = \mu D_r \frac{\partial \Omega}{\partial r} \quad (47)$$

$$\frac{\dot{\sigma}}{\sigma}\left(\ddot{\beta} + \dot{\beta}^2\right) = 0. \quad (48)$$

For the compressible case, $\dot{\sigma}/\sigma \neq 0$, so equation (48) leads to

$$\ddot{\beta} + \dot{\beta}^2 = 0. \quad (49)$$

In order to facilitate an analytic solution, consider again the case

$$\sigma(t) = \rho_0 e^{\int^t c(t')dt'}, \quad \dot{\alpha}(t) = -\frac{1}{2}a(t), \quad \dot{\beta}(t) = b(t). \quad (50)$$

Equations (41), (43), and (49) then yield a closed set of equations for the quantities $a(t)$, $b(t)$, and $c(t)$ (which are the same as those for the 2D case):

$$b(t) = \frac{1}{t + A}, \quad c(t) = \frac{e^{\frac{1}{2}\int^t a(t')dt'}}{B - \int^t e^{\frac{1}{2}\int^{t'} a(t'')dt''}dt'}, \quad a(t) = b(t) + c(t) \quad (51)$$

where A and B are arbitrary constants.

Equation (47) then becomes

$$\left(\rho_0 e^{\int^t c(t')dt'} - \frac{\rho_0}{U}cr\right)\left(\frac{\partial \Omega}{\partial t} - \frac{1}{2}ar\frac{\partial \Omega}{\partial r} - a\Omega\right) - \frac{\left(\frac{\rho_0}{U}\right)c}{\left(\rho_0 e^{\int^t c(t')dt'} - \frac{\rho_0}{U}cr\right)}\mu\frac{\partial \Omega}{\partial r} = \mu D_r \frac{\partial \Omega}{\partial r}. \quad (52)$$

In order to simplify equation (52), let us assume again that the compressibility effects are weak. (41) and (50) imply that the quantity $c(t)$, which is a measure of the density change, is then small. Equation (43) becomes

$$\frac{dc}{dt} - \frac{1}{2}a(t)c \approx 0 \quad (53)$$

from which

$$c(t) \approx c_0 e^{\frac{1}{2}\int^t a(t')dt'} \quad (54)$$

c_0 being an arbitrary constant. (54) replaces the second of the three solutions in (51). Further, in the weakly-compressible case, equation (48) (which, to first approximation, is automatically satisfied) does not lead to equation (49) which, therefore, has to be dropped. This implies that the first of the three solutions in (51), which comes from equation (49), also has to be dropped, as in the 2D case.

Thus, keeping only terms of $O(c)$, equation (52) may be approximated by

$$\frac{\partial \Omega}{\partial t} - \frac{\nu c}{U}\frac{\partial \Omega}{\partial r} - \frac{ar}{2}\frac{\partial \Omega}{\partial r} \approx a\Omega + \nu\left(1 + \frac{cr}{U}\right)D_r\frac{\partial \Omega}{\partial r} \quad (55)$$

where

$$\nu \equiv \frac{\mu}{\rho_0}.$$

Let us look for a solution of the form (à la Lundgren [18])

$$\Omega(r,t) = S(t)\hat{\Omega}(\xi,\tau)$$

$$\xi \equiv \sqrt{S(t)}\, r, \quad \tau \equiv \int_0^t S(t')dt', \quad S(t) = e^{\int_0^t a(t')dt'}. \quad (56)$$

Equation (55) then becomes

$$\frac{\partial \hat{\Omega}}{\partial \tau} - \hat{c}\frac{\partial \hat{\Omega}}{\partial \xi} \approx \nu\left(1 + \frac{\hat{c}}{\nu}\xi\right)D_\xi\frac{\partial \hat{\Omega}}{\partial \xi} \quad (57)$$

where

$$\hat{c}(t) \equiv \frac{\nu c(t)}{U\sqrt{S(t)}} \quad D_\xi \equiv \frac{\partial}{\partial \xi} + \frac{1}{\xi}.$$

The imposed straining flow has been transformed away by the Lundgren transformation (56). The approximation implicit in equation (56) gets better again as viscosity increases. Observe that the second term on the left hand side in equation (57) represents the *first*-order contribution due to the compressibility effects (assumed to be weak) — the compressibility effects impart a hyperbolic character to equation (57), as in the 2D case.

(i) Quasi-steady solution

Equation (57) admits a quasi-steady solution given by

$$\hat{\Omega} = \hat{\Omega}(\xi) \quad (58)$$

which satisfies

$$\hat{c}\frac{\partial \hat{\Omega}}{\partial \xi} + \nu\left(1 + \frac{\hat{c}}{\nu}\xi\right)D_\xi\frac{\partial \hat{\Omega}}{\partial \xi} \approx 0. \quad (59)$$

Equation (59) has the solution

$$\hat{\Omega} \approx CEi\left(\frac{\hat{c}}{\nu}\xi\right) = CEi\left(\frac{c(t)}{U}r\right) \tag{60}$$

where $Ei(x)$ is the exponential integral

$$Ei(x) \equiv \int_x^\infty \frac{e^u}{u}du,$$

and C is an arbitrary constant.

(60) has the following asymptotic behavior -

$$\hat{\Omega} \sim \frac{1}{r}e^{-\frac{c(t)}{U}r}, \quad r \text{ large.} \tag{61}$$

The exponential decay of the vorticity for large r signifies the enhanced compactness of the vortices due to an *externally-imposed* density build-up near the axis. One may in fact view (61) as a "*dressed*" vortex in analogy with the terminology in the dielectric screening of a charged particle polarizing the surrounding medium (Ashcroft and Mermin [26])! "*Dressed*" vortex owes its existence to a counter-conventional *externally-imposed* density build-up in the vortex core[3] (which is in contrast to a density drop in the vortex core in a normal compressible case).

(ii) Unsteady solution

For the unsteady case, equation (57) has an approximate solution

$$\hat{\Omega} \approx f(\xi + \hat{c}\tau)\frac{1}{\tau}e^{-\frac{\xi^2}{4\nu\tau}} \tag{62}$$

where $f(x)$ is an arbitrary function of x. (62) may be viewed as a propagating axisymmetric vortex in a compressible fluid[4].

If $a(t) = \text{const.} = a$,[5] using (62),(56) becomes

$$\Omega \approx a\frac{f\left[e^{\frac{1}{2}at}\left\{r + \frac{\nu c}{aU}\left(1 - e^{-at}\right)\right\}\right]}{\left(1 - e^{-at}\right)}\, e^{-\frac{ar^2}{4\nu(1-e^{-at})}}. \tag{63}$$

Observe that (63), in the limit $t \Rightarrow \infty$, gives the axisymmetric steady Burgers vortex:

$$\Omega \approx \frac{\Gamma a}{4\pi\nu}e^{-\frac{ar^2}{4\nu}} \tag{64}$$

[3] Such vortices do not appear to be stable because the density increase in a direction opposite to that of the effective gravity due to the centrifugal force (which is directed away from the axis) would correspond to a top heavy arrangement under gravity. Indeed, swirling flows are found to be stabilized by a density stratification increasing in the radial direction (Howard [27]) while the vortex breakdown process is found to be delayed by the latter type of density stratification (Shivamoggi and Uberoi [28]).

[4] Dissipative effects subsequently intervene and sustain the reconnection process.

[5] The case $a(t) = \text{const.}$, according to (51), is valid only in the weak-compressibility limit (small c), and in the generic situation (arbitrary c) it is not valid.

where Γ is the circulation around the vortex (and f has been chosen suitably).

The azimuthal velocity corresponding to (64) is

$$W = \frac{\Gamma}{2\pi r}\left(1 - e^{-\frac{ar^2}{4\nu}}\right). \tag{65}$$

(65) describes a rigid-body rotation for small r, and an irrotational flow field for large r. The azimuthal velocity is maximum for $r = r_* \sim \sqrt{\nu/a}$. Thus, r_*, which may be taken to be radius of the vortex core, is of the order of Kolmogorov microscale $\eta \sim (\nu^3/\epsilon)^{1/4}$, if $a \sim \sqrt{\epsilon/\nu}$ ϵ being the energy dissipation rate in turbulence.

5 Applications to turbulence

(i) Incompressible turbulence

(64) implies that the relevant length scale for the incompressible case is

$$\ell^2 \sim \frac{\nu}{a}. \tag{66}$$

Taking the core radii of Burgers vortices to be of the order of Kolmogorov microscale we have

$$a \sim \epsilon^{1/2}\nu^{-1/2} \tag{67}$$

where,

$$\epsilon \sim \nu\frac{U^2}{\ell^2}. \tag{68}$$

Equations (66)–(68) lead to

$$U \sim \epsilon^{1/3}\ell^{1/3} \tag{69}$$

and hence to the celebrated Kolmogorov [29] spectrum for incompressible turbulence

$$E(k) \sim \epsilon^{2/3}k^{-5/3}. \tag{70}$$

(ii) Compressible turbulence

(37) and (63) imply that the relevant length scale for the compressible case is

$$\ell \sim \frac{\nu c}{aU}. \tag{71}$$

Recalling that $c \Rightarrow 0$ corresponds to the incompressible limit, we may write

$$c \sim \frac{MU}{\ell} \tag{72}$$

where M is a reference Mach number of the flow

$$M \sim \frac{U}{C} \tag{73}$$

C being a reference speed of sound.

Further, writing

$$a \sim \frac{U}{\ell} \tag{74}$$

we have, from (71),

$$\ell \sim \frac{\nu}{C}. \tag{75}$$

On noting now that the energy dissipation rate can be written as

$$\hat{\epsilon} \sim \mu \frac{U^2}{\ell^2} \tag{76}$$

we obtain, from (75),

$$U \sim \rho^{-1/2} \hat{\epsilon}^{1/2} C^{-1/2} \ell^{1/2} \tag{77}$$

which leads to the Kadomtsev-Petviashvili [30] spectrum for compressible turbulence

$$E(k) \sim \hat{\epsilon} C^{-1} k^{-2}. \tag{78}$$

6 Vortex reconnection in a compressible fluid

We consider a generalization of Greene's [31] *local vortex pseudo-advection* (the terminology is, however, due to Vallis et al. [32]) to make a general discussion of the vortex reconnection process in a compressible fluid.

The vorticity evolution equation in an inviscid fluid is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = -\nabla \times \left(\frac{1}{\rho} \nabla p \right). \tag{79}$$

The term on the right represents *baroclinic* vorticity generation which is due to the misalignment of density and pressure gradients. Note that for an incompressible or a compressible barotropic fluid this term vanishes, so the vorticity evolution in an inviscid incompressible or a compressible barotropic fluid is simply a *local vortex advection* signifying the absence of vortex reconnection.

On the other hand, for a compressible non-barotropic fluid, if

$$\omega \cdot \frac{1}{\rho} \nabla p = 0, \quad \forall \mathbf{x} \in \mathcal{V} \tag{80}$$

$\mathcal{V}$ being the volume occupied by the fluid, i.e., the vortex lines are confined to the isobaric ($p =$ const.) surfaces, then one may write

$$\frac{1}{\rho} \nabla p = \boldsymbol{\omega} \times \mathbf{W}, \quad \forall \mathbf{x} \in \mathcal{V}. \tag{81}$$

Equation (79) then becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \times [\boldsymbol{\omega} \times (\mathbf{v} + \mathbf{W})] = 0. \tag{82}$$

(82) implies that, under condition (80), the vorticity evolution in a compressible non-barotropic fluid corresponds to a *local vortex pseudo-advection* by a modified velocity $\mathbf{v} + \mathbf{W}$ where, from (81),

$$\mathbf{W} = \frac{1}{\rho \, \omega^2} \nabla p \times \boldsymbol{\omega}. \tag{83}$$

Further, the helicity

$$H \equiv \boldsymbol{\omega} \cdot \mathbf{v} \tag{84}$$

which is a topological measure of the degree of knottedness of vortex lines, then evolves according to

$$\frac{\partial H}{\partial t} + \nabla \cdot [(\mathbf{v} + \mathbf{W}) H] = \nabla \cdot \left[\boldsymbol{\omega} \left(H + \frac{1}{2} v^2 \right) \right]. \tag{85}$$

Integrating equation (85) over the volume $\mathcal{V}(t)$ enclosed by a surface $S(t)$ moving with velocity $\mathbf{v} + \mathbf{W}$ on which $\omega \cdot \hat{\mathbf{n}} = 0$ (i.e., $S(t)$ is a vortex surface, as implied by equation (82)), we obtain

$$\frac{d}{dt} \int\limits_{\mathcal{V}(t)} H d\mathbf{x} = 0. \tag{86}$$

So, provided (80) is valid, the total helicity is conserved, even in a compressible non-barotropic fluid, despite the existence of *baroclinic* vorticity generation mechanism.

It should be noted however that the prevalence of *local vortex pseudo-advection* and hence the absence of vortex reconnection is a *sufficient* (but *not* necessary) condition for conserving the total helicity also in a compressible fluid. Therefore, the absence of *local vortex pseudo-advection* and hence the occurrence of vortex reconnection does not guarantee the destruction of the total helicity invariant.

In the generic compressible non-barotropic case, where (80) is not valid, the vorticity evolution does not correspond to a *local vortex pseudo-advection*. This paves the way for the occurrence of vortex reconnection in a compressible non-barotropic fluid even in the *inviscid* case! DNS of the reconnection process between two anti-parallel vortex tubes (Virk et al. [22]) in fact showed that *shocklet* formation was able to get reconnection going in a compressible fluid[6].

Inviscid compressible vortex reconnection is very akin to the *collisionless* magnetic reconnection process in high-temperature tenuous plasmas where resistivity is negligible (Coppi [33], Schindler [34], Drake and Lee [35], Ottaviani and Porcelli [36], Shivamoggi [37]–[39]). Here, the conservation of magnetic flux is replaced by the conservation of *generalized* magnetic flux (that now includes contributions from the electron-fluid momentum). So, magnetic flux changes and magnetic reconnection processes are sustainable even without resistivity!

7 Discussion

In this paper, stretched (modified Burgers) vortices are considered in a compressible fluid. The flows associated with the vortices are *perpendicular* to the plane of the uniform straining flows — a situation relevant for a mixing-layer flows. Compressibility effects have been restricted to be weak to facilitate analytic solutions. The compressible axisymmetric stretched vortex

[6] Dissipative effects subsequently intervene and sustain the reconnection process.

- exhibits exponential decay of the vorticity for large r signifying the enhanced compactness of the vortices caused by an *externally-imposed* density build-up near the axis – "dressed" vortices,
- has the axisymmetric Burgers vortex as the asymptotic limit ($t \Rightarrow \infty$).

The compressible vortex flow solutions in the 2D as well as axisymmetric cases identify a length scale relevant for the compressible case which leads to the Kadomtsev-Petviashvili [30] spectrum for compressible turbulence.

Vortex reconnection in a compressible non-barotropic fluid is possible even in the *inviscid* case – compressibility leads to defreezing of vortex lines in the fluid. This is very similar to the *collisionless* magnetic reconnection process in high-temperature tenuous plasmas.

The possibility of vortex reconnection in an *inviscid* fluid can raise some questions of principle (á la Taylor, as quoted in [36], for the *collisionless* magnetic reconnection process). Since the process is *reversible* one might wonder if the reconnection in such a system is only a transient phenomenon and if the vortex lines will eventually unreconnect. However, the essential presence of even a very small viscosity would inhibit the latter process.

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